or

$$\frac{-1}{4\sin A\sin B\sin C}\begin{vmatrix} \sin^2 A+\sin^2 B+\sin^2 C, & \sin^2 A+\sin^2 B+\sin^2 C, & \sin^2 A+\sin^2 B+\sin^2 C \\ 1, & 1, & 1 \\ 2\sin^2 A, & 2\sin^2 B, & 2\sin^2 C \end{vmatrix}.$$

But this is equal to zero since two rows are alike after dividing out $\sin^2 A + \sin^2 B + \sin^2 C$. It is to be noticed also that the above determinant is equal to zero for any values whatever of A, B, C provided only that two of them are alike.

Also solved by ELIJAH SWIFT, G. W. HARTWELL, H. POLISH, R. M. MATHEWS, H. L. AGARD, H. S. UHLER, CLIFFORD N. MILLS, W. W. BURTON, CARL A. W. STROM, A. M. KENYON, J. H. WEAVER, and A. H. WILSON.

CALCULUS.

387. Proposed by C. N. SCHMALL, New York City.

Show that the volume bounded by the cone $x^2 + y^2 = (a - z)^2$ and the planes $x = 0$, $x = z$, is $\frac{4}{9}a^3$.

I. SOLUTION BY A. M. HARDING, University of Arkansas.

The projection on the XY-plane of the curve of intersection of the cone and the plane $z = x$ is $y^2 = a^2 - 2ax$.

If we change this equation to polar coördinates we obtain

$$\rho = \frac{a}{1+\cos\theta} = \frac{a}{2}\sec^2\frac{\theta}{2}.$$

Hence,

$$\frac{v}{2} = \int_0^{\pi/2}\int_0^{a/2\sec^2\theta/2}\int_x^{a-\sqrt{x^2+y^2}} dz\cdot\rho d\rho d\theta = \int_0^{\pi/2}\int_0^{a/2\sec^2\theta/2}(a - \sqrt{x^2+y^2} - x)\rho d\rho d\theta$$

$$= \int_0^{\pi/2}\int_0^{a/2\sec^2\theta/2}(a\rho - \rho^2 - \rho^2\cos\theta)d\rho d\theta = \frac{a^3}{24}\int_0^{\pi/2}\sec^4\frac{\theta}{2}d\theta = \frac{a^3}{9}.$$

Hence the entire volume is $v = \frac{2}{9}a^3$.

II. SOLUTION BY GEO. W. HARTWELL, Hamline University.

If this volume is sliced parallel to the xy plane, the sections between $z = 0$ and $z = a/2$ will be segments of circles and from $z = a/2$ to $z = a$ semicircles.

Integrating then we have

$$\int_0^{a/2}\left[\tfrac{1}{2}\pi(a-z)^2 + z\sqrt{a^2-2az} - (a-z)^2\cos^{-1}\frac{z}{a-z}\right]dz + \int_{a/2}^{a}\tfrac{1}{2}\pi(a-z)^2dz$$

$$= \int_0^{a}\tfrac{1}{2}\pi(a-z)^2dz + \int_0^{a/2}\left[z\sqrt{a^2-2az} - (a-z)^2\cos^{-1}\frac{z}{a-z}\right]dz.$$

Integrating we have

$$[-\tfrac{1}{6}\pi(a-z)^3]_{z=0}^{z=a} + \left[-\frac{(2a^2+6az)(a^2-2az)^{3/2}}{30a^2} + \frac{(a-z)^3}{3}\cos^{-1}\frac{z}{a-z} - \frac{a^2\sqrt{a^2-2az}}{3}\right.$$

$$\left. + \frac{(2a^2+2az)\sqrt{a^2-2az}}{9} - \frac{(2a^2+2az+3z^2)\sqrt{a^2-2az}}{45}\right]_{z=0}^{z=(a/2)} = \tfrac{2}{9}a^3.$$

Also solved by H. L. AGARD, FRANK R. MORRIS, NORMAN ANNING, H. S. UHLER, and the PROPOSER.

388. Proposed by PAUL CAPRON, U. S. Naval Academy.

If $f(x, y) = 0$ represents a curve having a simple tangency to the axis of x at the origin, the